6.2 Differential Equations: Growth and Decay

- **Use separation of variables to solve a simple differential equation.**
- **Use exponential functions to model growth and decay in applied problems.**

Differential Equations

In Section 6.1, you learned to analyze the solutions visually of differential equations using slope fields and to approximate solutions numerically using Euler's Method. Analytically, you have learned to solve only two types of differential equations—those of the forms $y' = f(x)$ and $y'' = f(x)$. In this section, you will learn how to solve a more general type of differential equation. The strategy is to rewrite the equation so that each variable occurs on only one side of the equation. This strategy is called *separation of variables*. (You will study this strategy in detail in Section 6.3.)

EXAMPLE 1 Solving a Differential Equation

$$y' = \frac{2x}{y}$$ Original equation

$$yy' = 2x$$ Multiply both sides by y.

$$\int yy'\,dx = \int 2x\,dx$$ Integrate with respect to x.

$$\int y\,dy = \int 2x\,dx$$ $dy = y'\,dx$

$$\frac{1}{2}y^2 = x^2 + C_1$$ Apply Power Rule.

$$y^2 - 2x^2 = C$$ Rewrite, letting $C = 2C_1$.

So, the general solution is $y^2 - 2x^2 = C$.

REMARK You can use implicit differentiation to check the solution in Example 1.

When you integrate both sides of the equation in Example 1, you don't need to add a constant of integration to both sides. When you do, you still obtain the same result.

$$\int y\,dy = \int 2x\,dx$$

$$\frac{1}{2}y^2 + C_2 = x^2 + C_3$$

$$\frac{1}{2}y^2 = x^2 + (C_3 - C_2)$$

$$\frac{1}{2}y^2 = x^2 + C_1$$

Some people prefer to use Leibniz notation and differentials when applying separation of variables. The solution to Example 1 is shown below using this notation.

$$\frac{dy}{dx} = \frac{2x}{y}$$

$$y\,dy = 2x\,dx$$

$$\int y\,dy = \int 2x\,dx$$

$$\frac{1}{2}y^2 = x^2 + C_1$$

$$y^2 - 2x^2 = C$$

Exploration

In Example 1, the general solution of the differential equation is

$$y^2 - 2x^2 = C.$$

Use a graphing utility to sketch the particular solutions for $C = \pm 2$, $C = \pm 1$, and $C = 0$. Describe the solutions graphically. Is the following statement true of each solution?

The slope of the graph at the point (x, y) is equal to twice the ratio of x and y.

Explain your reasoning. Are all curves for which this statement is true represented by the general solution?

Growth and Decay Models

In many applications, the rate of change of a variable y is proportional to the value of y. When y is a function of time t, the proportion can be written as shown.

Rate of change of y — is — proportional to y.

$$\frac{dy}{dt} = ky$$

The general solution of this differential equation is given in the next theorem.

THEOREM 6.1 Exponential Growth and Decay Model

If y is a differentiable function of t such that $y > 0$ and $y' = ky$ for some constant k, then

$$y = Ce^{kt}$$

where C is the **initial value** of y, and k is the **proportionality constant. Exponential growth** occurs when $k > 0$, and **exponential decay** occurs when $k < 0$.

Proof

$y' = ky$	Write original equation.
$\frac{y'}{y} = k$	Separate variables.
$\int \frac{y'}{y}\,dt = \int k\,dt$	Integrate with respect to t.
$\int \frac{1}{y}\,dy = \int k\,dt$	$dy = y'\,dt$
$\ln y = kt + C_1$	Find antiderivative of each side.
$y = e^{kt}e^{C_1}$	Solve for y.
$y = Ce^{kt}$	Let $C = e^{C_1}$.

So, all solutions of $y' = ky$ are of the form $y = Ce^{kt}$. Remember that you can differentiate the function $y = Ce^{kt}$ with respect to t to verify that $y' = ky$.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

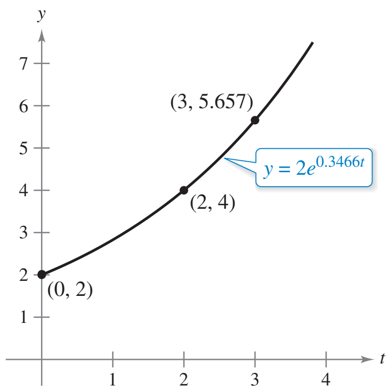
EXAMPLE 2 Using an Exponential Growth Model

The rate of change of y is proportional to y. When $t = 0$, $y = 2$, and when $t = 2$, $y = 4$. What is the value of y when $t = 3$?

Solution Because $y' = ky$, you know that y and t are related by the equation $y = Ce^{kt}$. You can find the values of the constants C and k by applying the initial conditions.

$2 = Ce^0$	⇨	$C = 2$	When $t = 0$, $y = 2$.
$4 = 2e^{2k}$	⇨	$k = \frac{1}{2}\ln 2 \approx 0.3466$	When $t = 2$, $y = 4$.

So, the model is $y = 2e^{0.3466t}$. When $t = 3$, the value of y is $2e^{0.3466(3)} \approx 5.657$ (see Figure 6.7).



If the rate of change of y is proportional to y, then y follows an exponential model.
Figure 6.7

Using logarithmic properties, the value of k in Example 2 can also be written as $\ln\sqrt{2}$. So, the model becomes $y = 2e^{(\ln\sqrt{2})t}$, which can be rewritten as $y = 2(\sqrt{2})^t$.

TECHNOLOGY Most graphing utilities have curve-fitting capabilities that can be used to find models that represent data. Use the *exponential regression* feature of a graphing utility and the information in Example 2 to find a model for the data. How does your model compare with the given model?

Radioactive decay is measured in terms of *half-life*—the number of years required for half of the atoms in a sample of radioactive material to decay. The rate of decay is proportional to the amount present. The half-lives of some common radioactive isotopes are listed below.

Uranium (^{238}U)	4,470,000,000 years
Plutonium (^{239}Pu)	24,100 years
Carbon (^{14}C)	5715 years
Radium (^{226}Ra)	1599 years
Einsteinium (^{254}Es)	276 days
Radon (^{222}Rn)	3.82 days
Nobelium (^{257}No)	25 seconds

EXAMPLE 3 Radioactive Decay

Ten grams of the plutonium isotope ^{239}Pu were released in a nuclear accident. How long will it take for the 10 grams to decay to 1 gram?

The Fukushima Daiichi nuclear disaster occurred after an earthquake and tsunami. Several of the reactors at the plant experienced full meltdowns.

Solution Let y represent the mass (in grams) of the plutonium. Because the rate of decay is proportional to y, you know that

$$y = Ce^{kt}$$

where t is the time in years. To find the values of the constants C and k, apply the initial conditions. Using the fact that $y = 10$ when $t = 0$, you can write

$$10 = Ce^{k(0)} \quad \Rightarrow \quad 10 = Ce^0$$

which implies that $C = 10$. Next, using the fact that the half-life of ^{239}Pu is 24,100 years, you have $y = 10/2 = 5$ when $t = 24,100$, so you can write

$$5 = 10e^{k(24,100)}$$

$$\frac{1}{2} = e^{24,100k}$$

$$\frac{1}{24,100} \ln \frac{1}{2} = k$$

$$-0.000028761 \approx k.$$

So, the model is

$$y = 10e^{-0.000028761t}. \qquad \text{Half-life model}$$

To find the time it would take for 10 grams to decay to 1 gram, you can solve for t in the equation

$$1 = 10e^{-0.000028761t}.$$

The solution is approximately 80,059 years.

REMARK The exponential decay model in Example 3 could also be written as $y = 10\left(\frac{1}{2}\right)^{t/24,100}$. This model is much easier to derive, but for some applications it is not as convenient to use.

From Example 3, notice that in an exponential growth or decay problem, it is easy to solve for C when you are given the value of y at $t = 0$. The next example demonstrates a procedure for solving for C and k when you do not know the value of y at $t = 0$.

KIMIMASA MAYAMA/EPA/Newscom

EXAMPLE 4 Population Growth

See LarsonCalculus.com for an interactive version of this type of example.

An experimental population of fruit flies increases according to the law of exponential growth. There were 100 flies after the second day of the experiment and 300 flies after the fourth day. Approximately how many flies were in the original population?

Solution Let $y = Ce^{kt}$ be the number of flies at time t, where t is measured in days. Note that y is continuous, whereas the number of flies is discrete. Because $y = 100$ when $t = 2$ and $y = 300$ when $t = 4$, you can write

$$100 = Ce^{2k} \quad \text{and} \quad 300 = Ce^{4k}.$$

From the first equation, you know that

$$C = 100e^{-2k}.$$

Substituting this value into the second equation produces the following.

$$\begin{aligned} 300 &= 100e^{-2k}e^{4k} \\ 300 &= 100e^{2k} \\ 3 &= e^{2k} \\ \ln 3 &= 2k \\ \frac{1}{2}\ln 3 &= k \\ 0.5493 &\approx k \end{aligned}$$

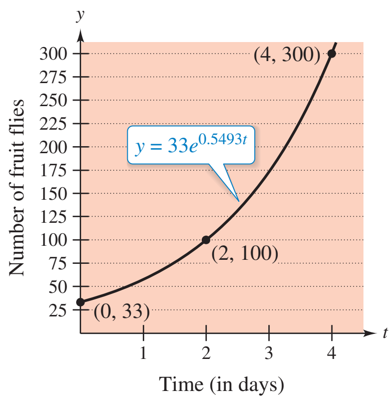


Figure 6.8

So, the exponential growth model is

$$y = Ce^{0.5493t}.$$

To solve for C, reapply the condition $y = 100$ when $t = 2$ and obtain

$$\begin{aligned} 100 &= Ce^{0.5493(2)} \\ C &= 100e^{-1.0986} \\ C &\approx 33. \end{aligned}$$

So, the original population (when $t = 0$) consisted of approximately $y = C = 33$ flies, as shown in Figure 6.8.

EXAMPLE 5 Declining Sales

Four months after it stops advertising, a manufacturing company notices that its sales have dropped from 100,000 units per month to 80,000 units per month. The sales follow an exponential pattern of decline. What will the sales be after another 2 months?

Solution Use the exponential decay model $y = Ce^{kt}$, where t is measured in months. From the initial condition $(t = 0)$, you know that $C = 100{,}000$. Moreover, because $y = 80{,}000$ when $t = 4$, you have

$$\begin{aligned} 80{,}000 &= 100{,}000e^{4k} \\ 0.8 &= e^{4k} \\ \ln(0.8) &= 4k \\ -0.0558 &\approx k. \end{aligned}$$

So, after 2 more months $(t = 6)$, you can expect the monthly sales rate to be

$$\begin{aligned} y &= 100{,}000e^{-0.0558(6)} \\ &\approx 71{,}500 \text{ units.} \end{aligned}$$

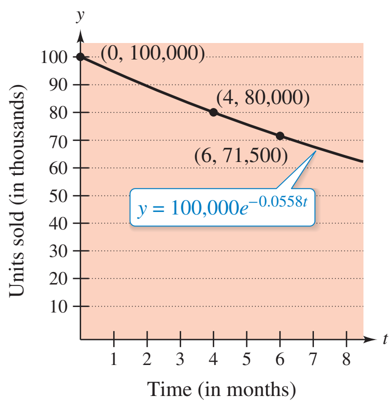


Figure 6.9

See Figure 6.9.

In Examples 2 through 5, you did not actually have to solve the differential equation $y' = ky$. (This was done once in the proof of Theorem 6.1.) The next example demonstrates a problem whose solution involves the separation of variables technique. The example concerns **Newton's Law of Cooling,** which states that the rate of change in the temperature of an object is proportional to the difference between the object's temperature and the temperature of the surrounding medium.

EXAMPLE 6 Newton's Law of Cooling

Let y represent the temperature (in °F) of an object in a room whose temperature is kept at a constant 60°. The object cools from 100° to 90° in 10 minutes. How much longer will it take for the temperature of the object to decrease to 80°?

Solution From Newton's Law of Cooling, you know that the rate of change in y is proportional to the difference between y and 60. This can be written as

$$y' = k(y - 60), \quad 80 \le y \le 100.$$

To solve this differential equation, use separation of variables, as shown.

$$\frac{dy}{dt} = k(y - 60) \qquad \text{Differential equation}$$

$$\left(\frac{1}{y - 60}\right) dy = k\,dt \qquad \text{Separate variables.}$$

$$\int \frac{1}{y - 60}\,dy = \int k\,dt \qquad \text{Integrate each side.}$$

$$\ln|y - 60| = kt + C_1 \qquad \text{Find antiderivative of each side.}$$

Because $y > 60$, $|y - 60| = y - 60$, and you can omit the absolute value signs. Using exponential notation, you have

$$y - 60 = e^{kt + C_1}$$

$$y = 60 + Ce^{kt}. \qquad C = e^{C_1}$$

Using $y = 100$ when $t = 0$, you obtain

$$100 = 60 + Ce^{k(0)} = 60 + C$$

which implies that $C = 40$. Because $y = 90$ when $t = 10$,

$$90 = 60 + 40e^{k(10)}$$

$$30 = 40e^{10k}$$

$$k = \frac{1}{10}\ln\frac{3}{4}.$$

So, $k \approx -0.02877$ and the model is

$$y = 60 + 40e^{-0.02877t}. \qquad \text{Cooling model}$$

When $y = 80$, you obtain

$$80 = 60 + 40e^{-0.02877t}$$

$$20 = 40e^{-0.02877t}$$

$$\frac{1}{2} = e^{-0.02877t}$$

$$\ln\frac{1}{2} = -0.02877t$$

$$t \approx 24.09 \text{ minutes.}$$

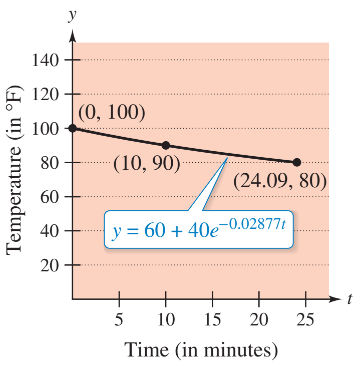


Figure 6.10

So, it will require about 14.09 *more* minutes for the object to cool to a temperature of 80° (see Figure 6.10).

6.2 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Solving a Differential Equation **In Exercises 1–10, solve the differential equation.**

1. $\dfrac{dy}{dx} = x + 3$

2. $\dfrac{dy}{dx} = 5 - 8x$

3. $\dfrac{dy}{dx} = y + 3$

4. $\dfrac{dy}{dx} = 6 - y$

5. $y' = \dfrac{5x}{y}$

6. $y' = -\dfrac{\sqrt{x}}{4y}$

7. $y' = \sqrt{x}\,y$

8. $y' = x(1 + y)$

9. $(1 + x^2)y' - 2xy = 0$

10. $xy + y' = 100x$

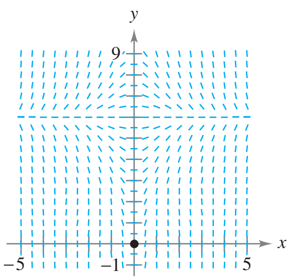
Writing and Solving a Differential Equation **In Exercises 11 and 12, write and solve the differential equation that models the verbal statement.**

11. The rate of change of Q with respect to t is inversely proportional to the square of t.

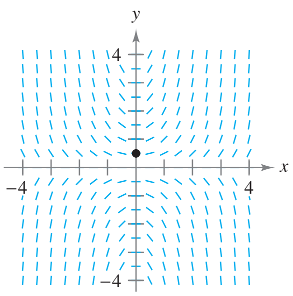
12. The rate of change of P with respect to t is proportional to $25 - t$.

Slope Field **In Exercises 13 and 14, a differential equation, a point, and a slope field are given. (a) Sketch two approximate solutions of the differential equation on the slope field, one of which passes through the given point. (b) Use integration to find the particular solution of the differential equation and use a graphing utility to graph the solution. Compare the result with the sketch in part (a). To print an enlarged copy of the graph, go to *MathGraphs.com*.**

13. $\dfrac{dy}{dx} = x(6 - y), \quad (0, 0)$



14. $\dfrac{dy}{dx} = xy, \quad \left(0, \dfrac{1}{2}\right)$



Finding a Particular Solution **In Exercises 15–18, find the function $y = f(t)$ passing through the point (0, 10) with the given first derivative. Use a graphing utility to graph the solution.**

15. $\dfrac{dy}{dt} = \dfrac{1}{2}t$

16. $\dfrac{dy}{dt} = -9\sqrt{t}$

17. $\dfrac{dy}{dt} = -\dfrac{1}{2}y$

18. $\dfrac{dy}{dt} = \dfrac{3}{4}y$

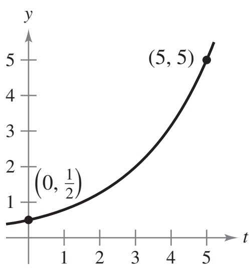
Writing and Solving a Differential Equation **In Exercises 19 and 20, write and solve the differential equation that models the verbal statement. Evaluate the solution at the specified value of the independent variable.**

19. The rate of change of N is proportional to N. When $t = 0$, $N = 250$, and when $t = 1$, $N = 400$. What is the value of N when $t = 4$?

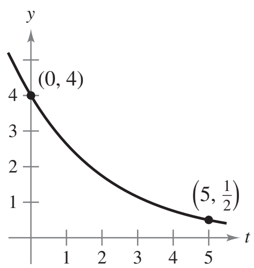
20. The rate of change of P is proportional to P. When $t = 0$, $P = 5000$, and when $t = 1$, $P = 4750$. What is the value of P when $t = 5$?

Finding an Exponential Function **In Exercises 21–24, find the exponential function $y = Ce^{kt}$ that passes through the two given points.**

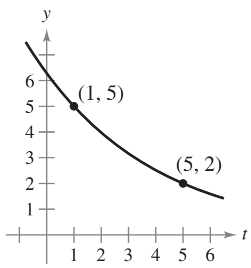
21.



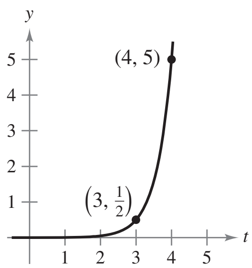
22.



23.



24.



WRITING ABOUT CONCEPTS

25. **Describing Values** Describe what the values of C and k represent in the exponential growth and decay model, $y = Ce^{kt}$.

26. **Exponential Growth and Decay** Give the differential equation that models exponential growth and decay.

Increasing Function **In Exercises 27 and 28, determine the quadrants in which the solution of the differential equation is an increasing function. Explain. (Do not solve the differential equation.)**

27. $\dfrac{dy}{dx} = \dfrac{1}{2}xy$

28. $\dfrac{dy}{dx} = \dfrac{1}{2}x^2y$

Radioactive Decay **In Exercises 29–36, complete the table for the radioactive isotope.**

	Isotope	Half-life (in years)	Initial Quantity	Amount After 1000 Years	Amount After 10,000 Years
29.	^{226}Ra	1599	20 g		
30.	^{226}Ra	1599		1.5 g	
31.	^{226}Ra	1599			0.1 g
32.	^{14}C	5715			3 g
33.	^{14}C	5715	5 g		
34.	^{14}C	5715		1.6 g	
35.	^{239}Pu	24,100		2.1 g	
36.	^{239}Pu	24,100			0.4 g

37. Radioactive Decay Radioactive radium has a half-life of approximately 1599 years. What percent of a given amount remains after 100 years?

38. Carbon Dating Carbon-14 dating assumes that the carbon dioxide on Earth today has the same radioactive content as it did centuries ago. If this is true, the amount of ^{14}C absorbed by a tree that grew several centuries ago should be the same as the amount of ^{14}C absorbed by a tree growing today. A piece of ancient charcoal contains only 15% as much of the radioactive carbon as a piece of modern charcoal. How long ago was the tree burned to make the ancient charcoal? (The half-life of ^{14}C is 5715 years.)

Compound Interest **In Exercises 39–44, complete the table for a savings account in which interest is compounded continuously.**

	Initial Investment	Annual Rate	Time to Double	Amount After 10 Years
39.	\$4000	6%		
40.	\$18,000	$5\frac{1}{2}$%		
41.	\$750		$7\frac{3}{4}$ yr	
42.	\$12,500		20 yr	
43.	\$500			\$1292.85
44.	\$6000			\$8950.95

Compound Interest **In Exercises 45–48, find the principal *P* that must be invested at rate *r*, compounded monthly, so that \$1,000,000 will be available for retirement in *t* years.**

45. $r = 7\frac{1}{2}\%, \quad t = 20$

46. $r = 6\%, \quad t = 40$

47. $r = 8\%, \quad t = 35$

48. $r = 9\%, \quad t = 25$

Compound Interest **In Exercises 49 and 50, find the time necessary for \$1000 to double when it is invested at a rate of *r* compounded (a) annually, (b) monthly, (c) daily, and (d) continuously.**

49. $r = 7\%$

50. $r = 5.5\%$

Population **In Exercises 51–54, the population (in millions) of a country in 2011 and the expected continuous annual rate of change *k* of the population are given. (*Source: U.S. Census Bureau, International Data Base*)**

(a) Find the exponential growth model

$$P = Ce^{kt}$$

for the population by letting $t = 0$ correspond to 2010.

(b) Use the model to predict the population of the country in 2020.

(c) Discuss the relationship between the sign of *k* and the change in population for the country.

	Country	2011 Population	k
51.	Latvia	2.2	-0.006
52.	Egypt	82.1	0.020
53.	Uganda	34.6	0.036
54.	Hungary	10.0	-0.002

55. Modeling Data One hundred bacteria are started in a culture and the number N of bacteria is counted each hour for 5 hours. The results are shown in the table, where t is the time in hours.

t	0	1	2	3	4	5
N	100	126	151	198	243	297

(a) Use the regression capabilities of a graphing utility to find an exponential model for the data.

(b) Use the model to estimate the time required for the population to quadruple in size.

56. Bacteria Growth The number of bacteria in a culture is increasing according to the law of exponential growth. There are 125 bacteria in the culture after 2 hours and 350 bacteria after 4 hours.

(a) Find the initial population.

(b) Write an exponential growth model for the bacteria population. Let t represent time in hours.

(c) Use the model to determine the number of bacteria after 8 hours.

(d) After how many hours will the bacteria count be 25,000?

57. Learning Curve The management at a certain factory has found that a worker can produce at most 30 units in a day. The learning curve for the number of units N produced per day after a new employee has worked t days is

$$N = 30(1 - e^{kt}).$$

After 20 days on the job, a particular worker produces 19 units.

(a) Find the learning curve for this worker.

(b) How many days should pass before this worker is producing 25 units per day?

58. Learning Curve Suppose the management in Exercise 57 requires a new employee to produce at least 20 units per day after 30 days on the job.

(a) Find the learning curve that describes this minimum requirement.

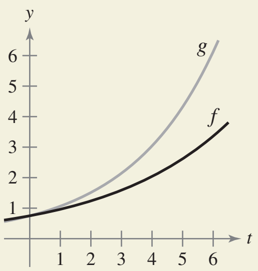
(b) Find the number of days before a minimal achiever is producing 25 units per day.

59. Insect Population

(a) Suppose an insect population increases by a constant number each month. Explain why the number of insects can be represented by a linear function.

(b) Suppose an insect population increases by a constant percentage each month. Explain why the number of insects can be represented by an exponential function.

60. HOW DO YOU SEE IT? The functions f and g are both of the form $y = Ce^{kt}$.



(a) Do the functions f and g represent exponential growth or exponential decay? Explain.

(b) Assume both functions have the same value of C. Which function has a greater value of k? Explain.

61. Modeling Data The table shows the resident populations P (in millions) of the United States from 1920 to 2010. (*Source: U.S. Census Bureau*)

Year	1920	1930	1940	1950	1960
Population, P	106	123	132	151	179

Year	1970	1980	1990	2000	2010
Population, P	203	227	249	281	309

(a) Use the 1920 and 1930 data to find an exponential model P_1 for the data. Let $t = 0$ represent 1920.

(b) Use a graphing utility to find an exponential model P_2 for all the data. Let $t = 0$ represent 1920.

(c) Use a graphing utility to plot the data and graph models P_1 and P_2 in the same viewing window. Compare the actual data with the predictions. Which model better fits the data?

(d) Use the model chosen in part (c) to estimate when the resident population will be 400 million.

62. Forestry

The value of a tract of timber is

$$V(t) = 100{,}000e^{0.8\sqrt{t}}$$

where t is the time in years, with $t = 0$ corresponding to 2010. If money earns interest continuously at 10%, then the present value of the timber at any time t is

$$A(t) = V(t)e^{-0.10t}.$$

Find the year in which the timber should be harvested to maximize the present value function.

63. Sound Intensity The level of sound β (in decibels) with an intensity of I is

$$\beta(I) = 10 \log_{10}\left(\frac{I}{I_0}\right)$$

where I_0 is an intensity of 10^{-16} watt per square centimeter, corresponding roughly to the faintest sound that can be heard. Determine $\beta(I)$ for the following.

(a) $I = 10^{-14}$ watt per square centimeter (whisper)

(b) $I = 10^{-9}$ watt per square centimeter (busy street corner)

(c) $I = 10^{-6.5}$ watt per square centimeter (air hammer)

(d) $I = 10^{-4}$ watt per square centimeter (threshold of pain)

64. Noise Level With the installation of noise suppression materials, the noise level in an auditorium was reduced from 93 to 80 decibels. Use the function in Exercise 63 to find the percent decrease in the intensity level of the noise as a result of the installation of these materials.

65. Newton's Law of Cooling When an object is removed from a furnace and placed in an environment with a constant temperature of 80°F, its core temperature is 1500°F. One hour after it is removed, the core temperature is 1120°F. Find the core temperature 5 hours after the object is removed from the furnace.

66. Newton's Law of Cooling A container of hot liquid is placed in a freezer that is kept at a constant temperature of 20°F. The initial temperature of the liquid is 160°F. After 5 minutes, the liquid's temperature is 60°F. How much longer will it take for its temperature to decrease to 30°F?

True or False? **In Exercises 67–70, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

67. In exponential growth, the rate of growth is constant.

68. In linear growth, the rate of growth is constant.

69. If prices are rising at a rate of 0.5% per month, then they are rising at a rate of 6% per year.

70. The differential equation modeling exponential growth is $dy/dx = ky$, where k is a constant.

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